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THE ROLE OF SYNTHETIC GEOMETRY IN REPRESENTATIONAL MEASUREMENT THEORY

by

UTA WILLE

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Sekr. MA 6-1, Straße des 17. Juni 136, D-10623 Berlin - Germany

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Uta Wille

1 Measurement and Geometry

Geometric representations of data and the formulation of quantitative models of observed phenomena are of main interest in all kinds of empirical sciences. To support the formulation of quantitative models, *representational measurement theory* studies the foundations of measurement. By mathematical methods it is analysed under which conditions attributes have numerical measurements and which numerical manipulations of the measurement values are meaningful (see Krantz et al. (1971)). In this paper, we suggest to discuss within the measurement theory approach both, the idea of geometric representations of data and the request to provide algebraic descriptions of dependencies of attributes. We show that, within such a broader paradigm of representational measurement theory, synthetic geometry can play a twofold role which enriches the theory and the possibilities of data interpretation.

In order to discuss the role of synthetic geometry in measurement theory, let us recall the basic idea of representational measurement theory. The representational approach first requires an exact description of the interface between reality and mathematics. Empirical data can, from an abstract point of view, be considered as a collection of relationships between specified objects. Therefore *empirical relational structures* are chosen as formal models describing the considered part of reality. Measurement is understood as the representation of empirical relational structures by *numerical relational structures* which are usually defined on the set of real numbers or an n-dimensional real vector space. The main problem of measurement theory is to formulate conditions which guarantee, for given empirical relational structures, the existence of homomorphisms into suitable numerical relational structures (*representation problem*).

Often there are different homomorphisms from a given empirical into a given numerical relational structure. Then the set of possible homomorphisms gives rise to the set of "admissible" transformations of the numerical relational structure

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which describe the connections between those homomorphisms. The admissible transformations express the degree of uniqueness of the representation. Thus, the characterization of the admissible transformations of the representing numerical relational structure is the second important task of measurement theory (*uniqueness problem*). Since geometry originates from the practical problem of measuring land it is natural to interpret the relationship of synthetic and analytic geometry within the measurement theory approach as follows (cf. Suppes et al. (1989) pp. 1-2):

> *"Historically, the earliest example of numerical representation was the invention of analytic geometry, which provides coordinate-vector representations for qualitative geometrical structures formulated in terms of such primitives as points, lines, comparative distances, and angles.*
> ...
> *There are, in fact, two quite distinct developments to be considered: analytic geometry, which formulates the spaces of numerical geometrical structures that potentially may serve to represent qualitative geometrical structures, and synthetic geometry, which develops the axiomatic theories of those qualitative structures. The pattern is the same as in measurement theory: a representation theorem shows how to embed a qualitative structure isomorphically into some family of numerical structures, and the corresponding uniqueness theorem describes the different ways that the embedding is possible."*

In this interpretation, synthetic geometric spaces (for instance: the affine, projective, or euclidean spaces) play the role of the empirical structures to be represented numerically. This is adequate in view of the history of geometry because the euclidean space has been constituted as a formal model of the visual space. On the other hand, geometric spaces are used in data analysis as representing structures into which empirical structures are embedded. Because geometric representations support the interpretation of data and help to reveal dependencies of attributes, measurement theory should consider geometric spaces as representing structures as well. This presupposes a broader notion of measurement which opens richer possibilities of interpretation since representability by more general structures requires weaker conditions for the underlying data. Instead of only considering numerical assignments, measurement may be understood more generally as the representation of empirical structures by synthetic or analytic (coordinatized) geometric spaces.

With other words, synthetic geometry can play an important role in representational measurement theory if we consider representations of empirical structures by geometric spaces in general. Representing synthetic geometric spaces may possess an algebraic description or not. Often it may be easier to constitute an geometric embedding into a synthetic geometric space which can be coordinatized by an algebraic structure than proving directly the embeddability into the algebraic structure. Furthermore, synthetic geometry can be useful if quantitative models are supposed to be found.

In this paper we try to verify this twofold role of synthetic geometry in representational measurement theory by considering different kinds of geometric representations of ordinal contexts. First, the classical case of representations in real vector spaces is considered in Section 2. Then it is shown in Section 3 how the split of the representation theorem into an embedding and a coordinatization theorem is useful in the case of ordered n-quasigroup representations. More generally, it is discussed in Section 4 which kinds of abstract geometric structures should be considered as representing structures. Finally, we conclude in the last section with the Four-Level-Paradigm suggested by this analysis of the role of synthetic geometry in representational measurement theory.

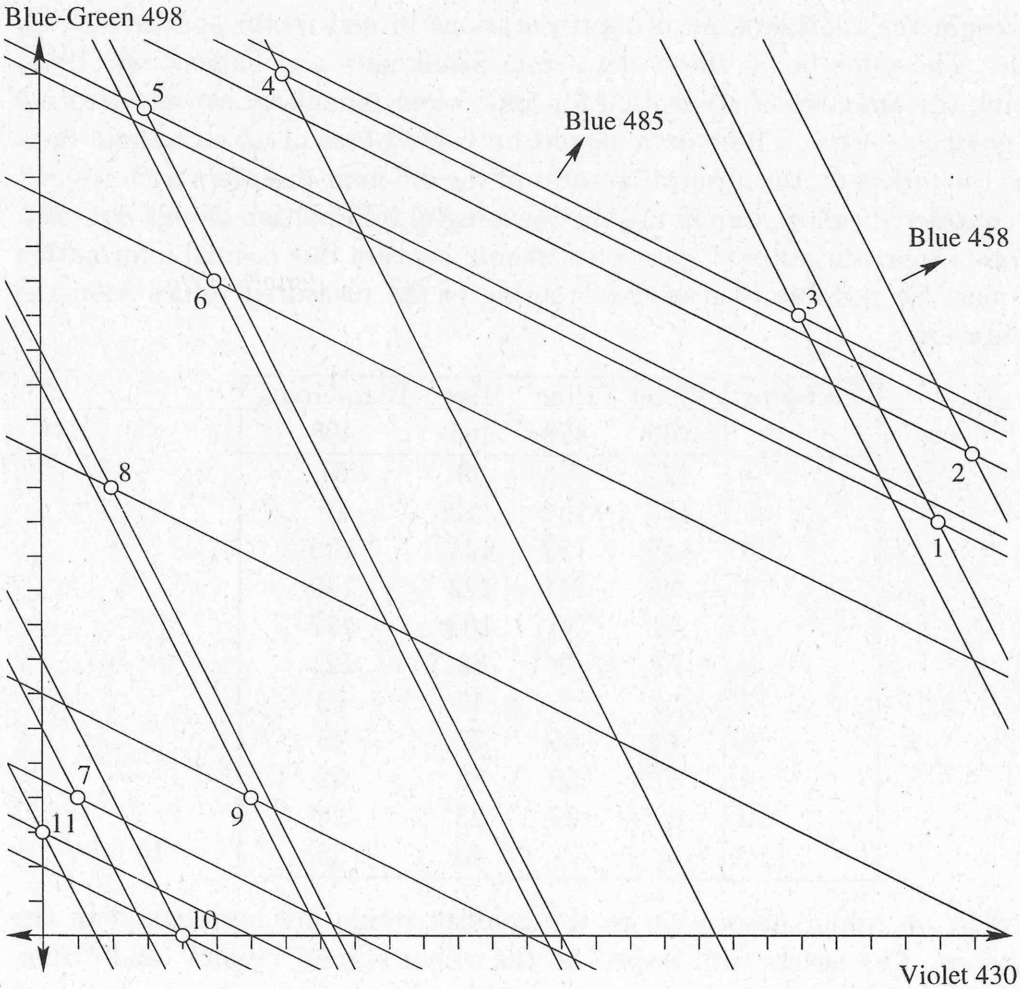
2 Representations in Real Vector Spaces

Let us begin the consideration of representations in real vector spaces with an example. The table below shows data from Schiffmann and Falkenberg (1968) describing the amounts of absorption for four colour stimuli by eleven receptors in the goldfish retina. These data should be viewed first of all as ordinal data because the orders on the absorption rates of the different receptors with respect to the considered colour stimuli are the meaningful information of this data set. Therefore a representation of these data should capture this ordinal information and it must be possible to read the ordering of the measured values from the representation.

Receptor	Violet 430	Blue 458	Blue 485	Blue-Green 498
1	147	153	89	57
2	153	154	110	75
3	145	152	125	100
4	99	101	122	140
5	46	85	103	127
6	73	78	85	121
7	14	2	46	52
8	44	65	77	73
9	87	59	58	52
10	60	27	23	24
11	0	0	40	39

These data on colour absorption in the goldfish retina are represented in the figure below. The orders with respect to the colour stimuli "Violet" and "Blue-Green 498" can be read from the representation by projecting to the first and second coordinate axis, respectively. The absorption rates of the colour stimuli "Blue 485" and "Blue 458" are represented by the two systems of parallel lines, where the lines represent the equivalence classes determined by the individual attribute values ordered in the direction of the little arrows. Here we have a representation as in additive conjoint measurement with the only difference that we have two dependent attributes instead of only one (cf. Chapter 6 and 9 of Krantz et al. (1971)). Now the crucial question is under which conditions such a simultaneous linear representation exists for more than three attributes in the plane.

To treat this representation problem, we choose ordinal contexts as empirical structures to be represented. Ordinal contexts formalize ordinal data tables and can be viewed as essentially the same mathematical model for ordinal data as relational structures usually considered in measurement theory (see Strahringer and R. Wille (1992)).



Definition 1 *A quadruple* $\mathbb{K} := (G, M, (W, \geq), I)$ *is called a* (complete) ordinal context *if* G *and* M *are sets,* $(W, \geq)$ *is a partially ordered set, and* I *is a ternary relation on* $G \times M \times W$ *such that, for each* $g \in G$ *and* $m \in M$*, there is exactly one element* w *in* W *with* $(g, m, w) \in I$*. The elements of* G*,* M*, and* W *are called* objects, attributes, *and* attribute values, *respectively.*

In the goldfish example the objects are the eleven receptors in the goldfish retina, the attributes are the considered colour stimuli, and the attribute values are the observed amounts of absorption. For ordinal contexts, one often writes $m(g) = w$ instead of $(g, m, w) \in I$ because the attribute m can also be understood as a map from G into W. Furthermore, $m(g) = w$ is read 'the object g has the value w for the attribute m'. After we have fixed a formal model of empirical data we now have to describe formally the idea of simultaneous linear representations.

Definition 2 *An ordinal context* $(G, M, (W, \geq), I)$ *with attributes* $m_1, \ldots, m_{n+l}$ *has a* simultaneous linear representation *in an* n*-dimensional real vector space with respect to* $m_1, \ldots, m_n$ *and real* n*-tuples* $\alpha_1^j, \ldots, \alpha_n^j$ *if there exists an injective mapping* $\varphi : G \longrightarrow \mathbb{R}^n$ *with*

$$m_s(g) \geq m_s(h) \quad \Leftrightarrow \quad \pi_s(\varphi(g)) \geq \pi_s(\varphi(h))$$

$$m_{n+j}(g) \geq m_{n+j}(h) \quad \Leftrightarrow \quad \sum_{s=1}^{n} \alpha_s^j \cdot \pi_s(\varphi(g)) \geq \sum_{s=1}^{n} \alpha_s^j \cdot \pi_s(\varphi(h))$$

for all $g, h \in G$*,* $s = 1, \ldots, n$*, and* $j = 1, \ldots, l$*, and* π_s *denoting the projection onto the* s*-th coordinate axis.*

I.e., in the case of simultaneous linear representations, the objects of an ordinal context are mapped into a real vector space such that the orders of n distinguished attributes are represented on the n coordinate axes and the orders of the remaining attributes can be obtained by different linear combinations of the coordinates. Thereby, the different linear combinations are specified by the n-tuples $\alpha_1^j, \ldots, \alpha_n^j$. Now, the question on the linear representability can be formulated as follows: *Under which conditions does an ordinal context have a simultaneous linear representation in a real vector space?*

For the additive representation of finitely many objects and $n+1$ attributes in an n-dimensional real vector space (i.e. for $l = 1$), Dana Scott (1964) gave a first characterization by necessary and sufficient conditions. It turns out that Scott's result can be generalized to simultaneous linear representations and the following representation theorem relies upon this generalization of Scott's theorem (see U. Wille (1995)).

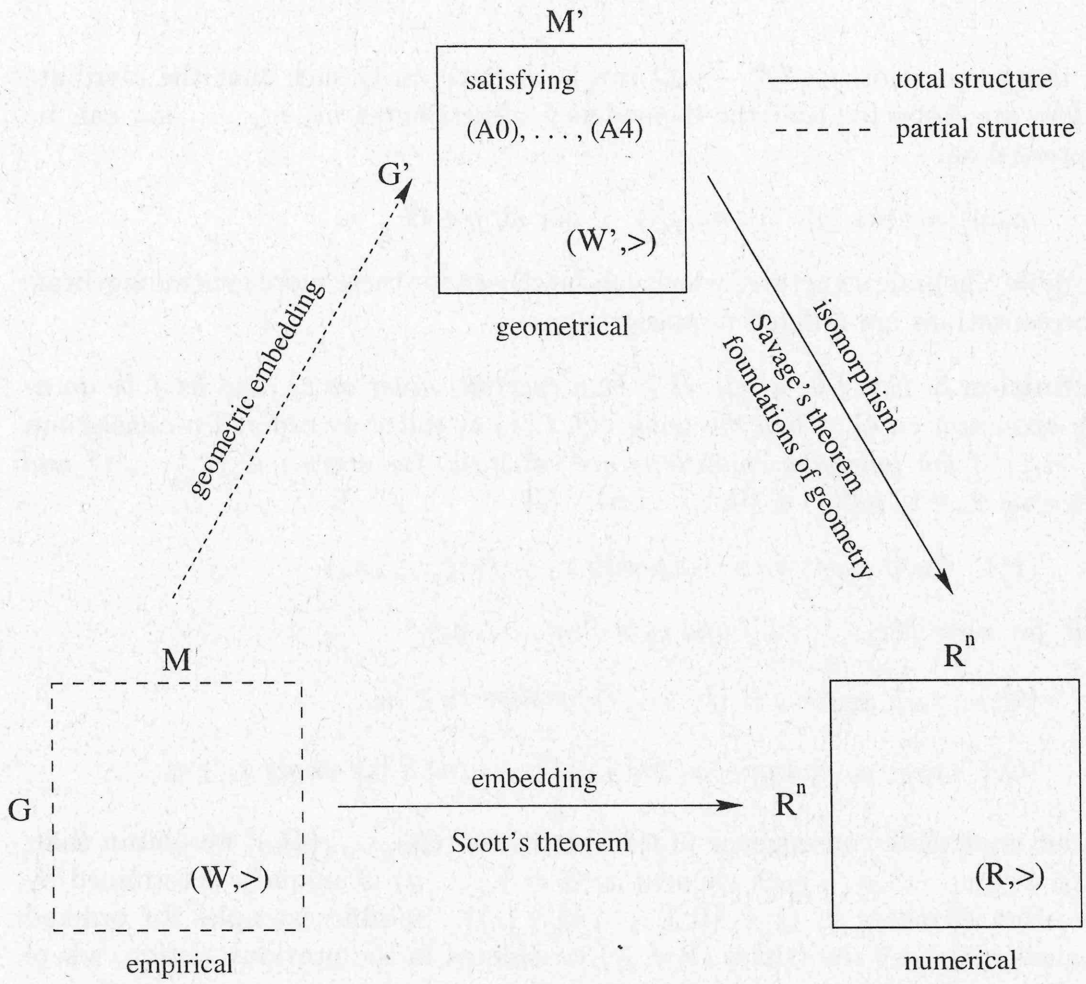
Theorem 1 (real representation theorem) *Let $\mathbb{K} := (G, M, (W, \geq), I)$ be a finite ordinal context with attributes $m_1, \ldots, m_{n+l}$ for which $\geq$ is linear on $m_i(G)$ $(i = 1, \ldots, n+l)$ and, for $j = 1, \ldots, l$, let $\alpha_1^j, \ldots, \alpha_n^j$ be n-tuples of nonnegative integers. Then $\mathbb{K}$ has a simultaneous linear representation in an n-dimensional real vector space with respect to $m_1, \ldots, m_n$ and $\alpha_1^j, \ldots, \alpha_n^j$ if and only if the following condition holds:*

> *If $u_j : G \longrightarrow \mathbb{I}$ ($\mathbb{I}$ the set of integers, $j = 1, \ldots, l$) are mappings with $\sum_{g \in G} u_j(g) = 0$ for $j = 1, \ldots, l$, $\sum_{m_{n+j}(g) \leq v} u_j(g) \leq 0$ for all $v \in m_{n+j}(G)$, $j = 1, \ldots, l$, and $\sum_{m_s(g) \leq w} \sum_{j=1}^{l} \alpha_s^j \cdot u_j(g) \geq 0$ for all $w \in m_s(G)$, $s = 1, \ldots, n$, then $\sum_{m_{n+j}(g) \leq v} u_j(g) = 0$ for all $v \in m_{n+j}(G)$, $j = 1, \ldots, l$, and $\sum_{m_s(g) \leq w} \sum_{j=1}^{l} \alpha_s^j \cdot u_j(g) = 0$ for all $w \in m_s(G)$, $s = 1, \ldots, n$.*

An even more technical version of this representation theorem holds if only partial orders on $m_i(G)$ are assumed. Note that in this representation theorem the n-tuples $\alpha_1^j, \ldots, \alpha_n^j$ are presumed (i.e., the slopes of the representing systems of parallel hyperplanes). Therefore it should be further clarified when there exist such n-tuples $\alpha_1^j, \ldots, \alpha_n^j$ and what are specific conditions for $\alpha_1^j, \ldots, \alpha_n^j$ needed to obtain bilinear representations.

The representation theorem shows that it is not easy to guarantee the existence of an embedding of a partial structure into a numerical structure. Often it is easier to find representation theorems if we want to obtain a total numerical structure as the image of the representation. I.e., in the goldfish example, if we want to obtain all points of the plane and all systems of parallel lines. There are many famous examples of theorems which characterize total numerical structures; just one example is Savage's expected utility theorem (cf. Savage (1954)). For ordinal contexts, such a representation theorem is provided in Chapter 3 of U. Wille (1995).

In general such representation theorems are studied in foundations of geometry constituting the interplay between synthetic and analytic geometry. Since there exists a rich knowledge on algebraic coordinatizations of synthetic geometric spaces one should also ask for geometric embeddings of ordinal contexts into synthetic geometric spaces. In the case of real representations it seems to be difficult to construct such geometric embeddings but, if we choose a more general algebraic structure as representing structure, the problem of geometric embedding and coordinatization can be solved. In the next section we report on such embeddings and coordinatizations. (For an overview see also the figure on the next page).



3 Representations by Ordered n-Quasigroups

Representations in real vector spaces provide effective algebraic descriptions of dependencies between attributes. Dependencies between attributes $m_0, m_1, \ldots, m_n$ are described algebraically by

$$m_0(g) = \alpha_1 \cdot m_1(g) + \ldots + \alpha_n \cdot m_n(g) \qquad \text{for all } g \in G,$$

where the attribute values of $m_0, m_1, \ldots, m_n$ are identified with real numbers such that the order relations on the attribute values are respected. Unfortunately, real representations require strong conditions for the underlying data in order to be representable. Since data, especially in social and behavioural sciences, do usually not fulfill such strong conditions, one should try to describe attribute dependencies by more general algebraic operations.

The idea again is to identify the attribute values of an ordinal context $\mathbb{K}$ with elements of a suitable ordered set $(Q, \geq)$ and to formulate conditions under which an n-ary operation $f : Q^n \longrightarrow Q$ can be defined on Q such that the attribute orders are respected and the dependency of attributes $m_0, m_1, \ldots, m_n$ can be expressed as

$$m_0(g) = f(m_1(g), \ldots, m_n(g)) \qquad \text{for all } g \in G.$$

Suitable algebraic structures which can be chosen for these more general algebraic representations are ordered n-quasigroups.

Definition 3 *Let Q be a set, let $\geq$ be a (partial) order on Q, and let f be an n-ary operation on Q. Then the triple $(Q, f, \geq)$ is called an* ordered n-quasigroup *$(n \geq 2)$ if the following conditions are satisfied: for every $i \in \{1, \ldots, n\}$ and elements $x_j \in Q$ with $j \in \{0, 1, \ldots, n\} \setminus \{i\}$*

(P_i) *there exists an $x_i \in Q$ with $x_0 = f(x_1, \ldots, x_n)$*

and, for $x_0 = f(x_1, \ldots, x_n)$ and $y_0 = f(y_1, \ldots, y_n)$,

(Q_0) *$x_j \geq y_j$ for $j \in \{1, \ldots, n\}$ implies $x_0 \geq y_0$,*

(Q_i) *$x_0 \geq y_0$ and $y_j \geq x_j$ for $j \in \{1, \ldots, n\} \setminus \{i\}$ imply $x_i \geq y_i$.*

As an immediate consequence of the conditions $(Q_0), \ldots, (Q_n)$, we obtain that, if $x_0 = f(x_1, \ldots, x_n)$, each element x_i $(i = 1, \ldots, n)$ is uniquely determined by the other elements x_j $(j \in \{0, 1, \ldots, n\} \setminus \{i\})$. Specific examples for ordered n-quasigroups are the triples $(\mathbb{R}, f, \geq)$ considered in the previous section, where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is the linear combination $f(x_1, \ldots, x_n) := \alpha_1 \cdot x_1 + \ldots + \alpha_n \cdot x_n$ with positive scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$.

This suggests to define representations by ordered n-quasigroups analogously to the linear case. For simplicity, we consider representations with only one dependent attribute; the general case is treated in U. Wille (1996). An ordinal context $\mathbb{K} := (G, M, (W, \geq), I)$ with $M := \{m_0, m_1, \ldots, m_n\}$ is said to be *representable by an ordered n-quasigroup* $(Q, f, \geq)$ if there exits an injective mapping $\varphi : G \longrightarrow Q^n$ with

$$m_s(g) \geq m_s(h) \quad \Leftrightarrow \quad \pi_s(\varphi(g)) \geq \pi_s(\varphi(h))$$

$$m_0(g) \geq m_0(h) \quad \Leftrightarrow \quad f(\varphi(g)) \geq f(\varphi(h))$$

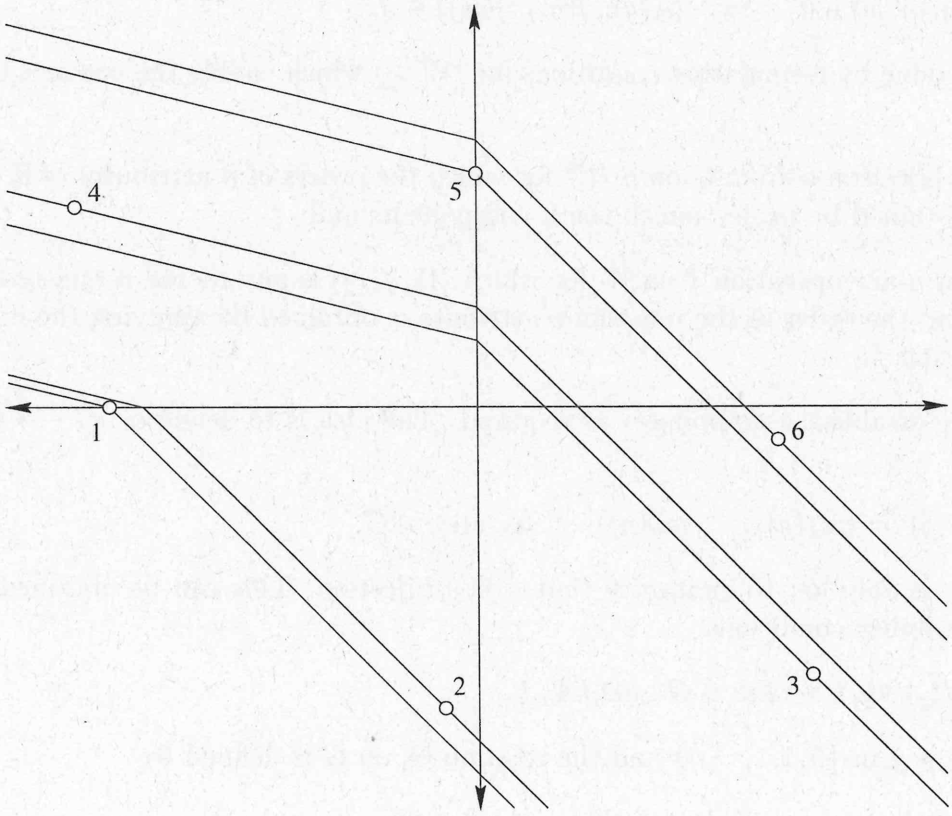
for all $g, h \in G$ and $s = 1, \ldots, n$. Before we discuss conditions for the representability of ordinal contexts by ordered n-quasigroups, we show that ordered n-quasigroup representations are indeed more general than linear ones.

The ordinal context $\mathbb{K}_6$ consisting of only six objects and three attributes does not have a linear representation in the euclidean plane which can be easily checked (cf. U. Wille (1995) Chapter 2). But $\mathbb{K}_6$ has a representation by the ordered quasigroup $(\mathbb{R}, \oplus, \geq)$ on the ordered set of real numbers with $\oplus$ defined by

$\mathbb{K}_6$	m_0	m_1	m_2
1	1	4	2
2	2	1	3
3	3	2	6
4	4	5	1
5	5	6	4
6	6	3	5

$$x \oplus y := \begin{cases} 1/4 \cdot x + y & \text{for } x < 0, y > 0, 1/4 \cdot x + y > 0 \\ x + 4 \cdot y & \text{for } x < 0, y > 0, 1/4 \cdot x + y \leq 0 \\ x + y & \text{otherwise.} \end{cases}$$

$(\mathbb{R}, \oplus, \geq)$ is even an ordered loop; i.e., it furthermore has an neutral element 0 with $x \oplus 0 = 0 \oplus x = 0$ for all $x \in \mathbb{R}$. The ordinal context $\mathbb{K}_6$ is visualized in the following diagram.



A representation theorem for ordered n-quasigroups is obtained in two steps. First the total image structure for such representations is characterized axiomatically and subsequently it is discussed when an ordinal context can be embedded into the obtained synthetic (axiomatically characterized) structure. This will be outlined without proofs; detailed proofs can be found in R. Wille and U. Wille (1992) and, for the case of more than one dependent attribute, in U. Wille (1996).

The total image structures for ordered n-quasigroup representations are the ordinal contexts $\mathbb{K}(Q, f, \geq) := (Q^n, \{f, \pi_1, \ldots, \pi_n\}, (Q, \geq), J)$, where $(Q, f, \geq)$ is an ordered n-quasigroup and J is defined by

$$((x_1, \ldots, x_n), p, q) \in J \quad :\Leftrightarrow \quad p(x_1, \ldots, x_n) = q$$

for $(x_1, \ldots, x_n) \in Q^n, p \in \{f, \pi_1, \ldots, \pi_n\}$, and $q \in Q$. Now, conditions have to be formulated under which an arbitrary ordinal context $\mathbb{K} := (G, M, (W, \geq), I)$ with $n+1$ attributes $m_0, m_1, \ldots, m_n$ is isomorphic to some $\mathbb{K}(Q, f, \geq)$; i.e., conditions under which there are bijections $\alpha : G \longrightarrow Q^n$, $\beta : M \longrightarrow \{f, \pi_1, \ldots, \pi_n\}$, and an order-isomorphism $\gamma : (W, \geq) \longrightarrow (Q, \geq)$ such that, for all $g \in G$, $m \in M$, and $w \in W$,

$$(g, m, w) \in I \quad \Leftrightarrow \quad (\alpha(g), \beta(m), \gamma(w)) \in J.$$

This is done by formulating conditions for $(W, \geq)$ which enable the construction of

1. a bijection α from G onto W^n for which the orders of n attributes of $\mathbb{K}$ are obtained by projection to the n components and

2. an n-ary operation f on W for which $(W, f, \geq)$ is an ordered n-quasigroup and the order of the remaining attribute is obtained by applying the operation f.

First we establish a mapping α as required. The idea is to define $\alpha : G \longrightarrow W^n$ by

$$\alpha(g) := (m_1(g), \ldots, m_n(g)) \qquad \text{for all } g \in G.$$

Then it is only left to guarantee that α is a bijection. This can be obtained by the solvability conditions

$$(P_{ij})\ \forall g, k \in G \exists h \in G : g\Theta_i h \Psi_{ij} k,$$

where $i \neq j$ in $\{0, 1, \ldots, n\}$ and the relation Θ_i on G is defined by

$$g\Theta_s h \quad \Leftrightarrow \quad m_s(g) = m_s(h) \qquad (g, h \in G)$$

for $s \in \{0, 1, \ldots, n\}$ and $\Psi_{ij} := \bigcap_{s \in \{0,1,\ldots,n\} \setminus \{i,j\}} \Theta_s$. An ordinal context $\mathbb{K}$ satisfying (P_{ij}) for all $i, j \in \{0, 1, \ldots, n\}$ with $i \neq j$ is called *solvable*.

For instance, the condition (P_{10}) yields, for objects $g, k \in G$ with $\alpha(g) = (v_1, \ldots, v_n)$ and $\alpha(k) = (w_1, \ldots, w_n)$, the existence of an object $h \in G$ with $\alpha(h) = (v_1, w_2 \ldots, w_n)$, where h is unique if $\Theta_1 \cap \ldots \cap \Theta_n = id_G$. For an ordinal context $\mathbb{K}$ with $n + 1$ attributes $m_0, m_1, \ldots, m_n$ and $m_i(G) = W$, it can be proved that the mapping α is bijective, if the conditions (P_{i0}) are satisfied for $i = 1, \ldots, n$ and $\Theta_1 \cap \ldots \cap \Theta_n = id_G$.

The remaining solvability conditions are needed in order to construct an n-quasigroup operation f on W. To make this plausible, we consider the ordinal context $\mathbb{K}(\mathbb{R}, f, \geq)$, where $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined by $f(x_1, x_2) = x_1 + x_2$. Then (P_{12}) yields that, for $(x_1, x_2), (z_1, z_2) \in \mathbb{R}^2$, there exists an $y \in \mathbb{R}$ with $x_1 + y = z_1 + z_2$. The uniqueness of the 'solution' can be obtained by $\Theta_0 \cap \Theta_1 = id_{\mathbb{R}^2}$. Therefore, we need besides the conditions (P_{ij}) that $\bigcap_{s \neq i} \Theta_s = id_G$ for $i = 0, 1, \ldots, n$.

Finally, the n-quasigroup operation f constructed on W has to respect the order on W; i.e., it has to satisfy the axioms $(Q_0), (Q_1), \ldots, (Q_n)$ of an ordered n-quasigroup. Therefore, corresponding ordinal dependencies have to be required for the attributes $m_0, m_1, \ldots, m_n$. To introduce ordinal dependency and to motivate the axioms (Q_i), let us again consider the classical case of additive representations by real numbers where $m_0(g) = m_1(g) + \ldots + m_n(g)$. For the validity of this equation it is obviously necessary that, if the values of the attributes $m_1, \ldots, m_n$ decrease while changing from one object to another, the value of m_0 has to decrease too. This observation is formalized in the following *ordinal dependency* axioms:

(A_0) $m_j(g) \geq m_j(h)$ for all $j \in \{1, \ldots, n\}$ implies $m_0(g) \geq m_0(h)$,

(A_i) $m_0(g) \geq m_0(h)$ and $m_j(h) \geq m_j(g)$ for $j \in \{1, \ldots, n\} \setminus \{i\}$ imply $m_i(g) \geq m_i(h)$

The conditions $(A_0), (A_1), \ldots, (A_n)$ are fundamental for the representability of ordinal contexts by ordered algebraic structures and they are necessary for the representability of ordinal contexts by ordered n-quasigroups. Furthermore, the conditions $(A_0), \ldots, (A_n)$ imply that $\bigcap_{j \in \{0,1,\ldots,n\} \setminus \{i\}} \Theta_j \subseteq \Theta_i$ for $i = 0, 1, \ldots, n$; i.e., $\bigcap_{j \in \{0,1,\ldots,n\} \setminus \{i\}} \Theta_j = \bigcap_{j=1}^{n} \Theta_j =: \Delta$. Therefore, the ordinal context $\mathbb{K}$ can be factored by the equivalence relation Δ without loss of structural information. It can be proved that the conditions (P_{ij}) together with $(A_0), (A_1), \ldots, (A_n)$ already characterize the ordinal context $\mathbb{K}(Q, f, \geq)$ if $\bigcap_{s \in \{0,1,\ldots,n\}} \Theta_s = id_G$ is assumed.

Theorem 2 (characterization of $\mathbb{K}(Q, f, \geq)$) *An ordinal context $\mathbb{K}$ with attribute set $\{m_0, m_1, \ldots, m_n\}$ is isomorphic to an ordinal context $\mathbb{K}(Q, f, \geq)$ for some ordered n-quasigroup $(Q, f, \geq)$ if and only if $\mathbb{K}$ is solvable with $\Theta_0 \cap \Theta_1 \cap \ldots \cap \Theta_n = id_G$ and satisfies the ordinal dependency conditions (A_i) for $i = 0, 1, \ldots, n$.*

An solvable ordinal context $\mathbb{K}$ with attribute set $\{m_0, m_1, \ldots, m_n\}$ satisfying $(A_0), (A_1), \ldots, (A_n)$ and $\Theta_0 \cap \Theta_1 \cap \ldots \cap \Theta_n = id_G$ is called an *ordered $(n+1)$-net*. I.e., ordered $(n+1)$-nets are the synthetic geometric structures which can be coordinatized by ordered n-quasigroups. The corresponding analytic geometric structures are the ordinal contexts $\mathbb{K}(Q, f, \geq)$. The following embedding theorem states that the ordinal dependency conditions $(A_0), (A)_1), \ldots, (A_n)$ already guarantee the existence of a geometric embedding of an arbitrary ordinal context into an ordered $(n+1)$-net. Thereby the geometric embedding can be constructed canonically by extending the underlying ordinal context step by step.

Theorem 3 (embedding theorem) *An ordinal context $\mathbb{K}$ satisfying the ordinal dependency conditions $(A_0), (A_1), \ldots, (A_n)$ can always be embedded into an ordered $(n+1)$-net.*

This means that an ordinal context $\mathbb{K}$ with attribute set $\{m_0, m_1, \ldots, m_n\}$ is representable by an ordered n-quasigroup if and only if its attributes satisfy (A_i) for $i = 0, 1, \ldots, n$. Furthermore, the ordered n-quasigroup used for the coordinatization can be transformed into an isotopic ordered n-loop which even provides us with an n-loop representation. The uniqueness of coordinatizations of ordinal contexts by ordered n-quasigroups is treated in R. Wille and U. Wille (1995).

4 Representations by Abstract Geometries

It was proposed in the first section to consider representations of empirical structures by geometric spaces in general. Up to now, it has not been discussed what kind of structures are meant by geometric spaces. From the standpoint of data analysis it seems useful to admit very general structures as representing structures because for the representability into more general structures only weaker conditions are necessary. For example, empirical data can only seldom be represented in a euclidean space, but there is a good chance for a representation in a more general geometric space. This suggests choosing a very general notion of geometric spaces for representations. Such a notion is given by the abstract geometries as defined by Maeda (1951). An abstract geometry consists of a set of points together with an operator which assigns to every subset of points a subspace generated by those points. Such a geometry can also be described as a set of points with a closure system of subspaces having particular properties (see also Jónsson (1959)).

Formally, an *abstract geometric space* is an ordered pair $(S, \mathcal{A})$ consisting of a set S and a set $\mathcal{A}$ of subsets of S with the following properties:

(i) S and $\emptyset$ are members of $\mathcal{A}$.

(ii) Every one-element subset of S is in $\mathcal{A}$.

(iii) The intersection of sets belonging to $\mathcal{A}$ is again in $\mathcal{A}$.

The elements of S and $\mathcal{A}$ are called *points* and *subspaces* of the geometric space $(S, \mathcal{A})$, respectively. A pair $(S, \mathcal{A})$ with $S \in \mathcal{A}$ satisfying (iii) is called a *closure system.* By (iii), we can associate to every set of points X a subspace $< X >:= \bigcap_{X \subseteq Y \in \mathcal{A}} Y$ 'generated' by X. Hence one could even choose more generally a set of points together with a closure system of subspaces as representing structure. A closure system still provides a notion of dependency. This is important because the identification of dependencies between attributes is one of the main interests in data analysis.

Let us again consider an ordinal context $\mathbb{K} := (G, M, (W, \geq), I)$. To $\mathbb{K}$ one can, for example, associate an 'ordinal space' $(G, \bigcup_{m \in M} \mathcal{S}_m)$, where $\mathcal{S}_m$ is defined by

$$\mathcal{S}_m := \{\{g \in G | m(g) \leq w\} \ | w \in m(G)\}.$$

Then we have, for $m \in M$ and $v, w \in m(G)$,

$$v \geq w \quad \Leftrightarrow \quad \{g \in G | m(g) \leq v\} \supseteq \{g \in G | m(g) \leq w\}$$

and $(G, \bigcup_{m \in M} \mathcal{S}_m)$ can be extended to a closure system $(G, \Gamma(\bigcup_{m \in M} \mathcal{S}_m))$ generated by $\bigcup_{m \in M} \mathcal{S}_m$. Therefore, every ordinal context $(G, M, (W, \geq), I)$ trivially can be represented by the ordinal context $(G, M, (\Gamma(\bigcup_{m \in M} \mathcal{S}_m), \supseteq), J)$ with $m(g) := \bigcap\{X \in \mathcal{S}_m | g \in X\}$. Note, that the definition of $\mathcal{S}_m$ as above is only one possibility. The definition of $\mathcal{S}_m$ always reflects the interpretation of the attribute values which should be further developed elsewhere.

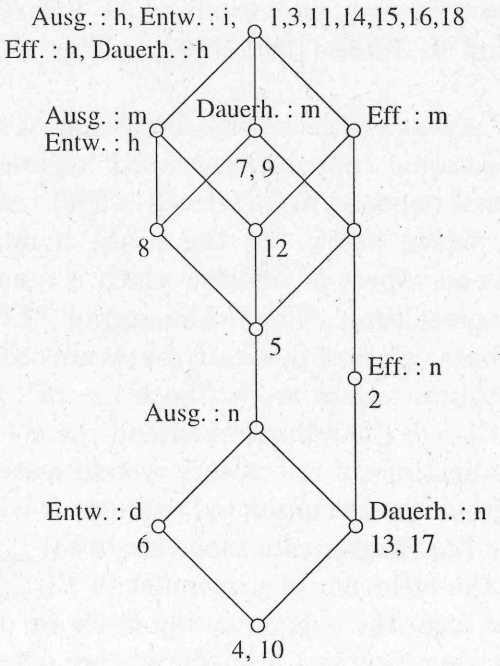
Now representations by closure systems are especially interesting if they have a visualization. For instance, one could try to visualize the obtained closure system by a Venn-diagram; i.e., try to identify the points and subspaces of the closure system with points and discs and their intersections in the euclidean plane such that set inclusion is respected. Furthermore, the general approach of representations by closure systems has successfully been activated in formal concept analysis, where closure systems are visualized by line diagrams. (cf. R. Wille (1982) and Ganter and R. Wille (1989, 1996)). This also supports our general approach.

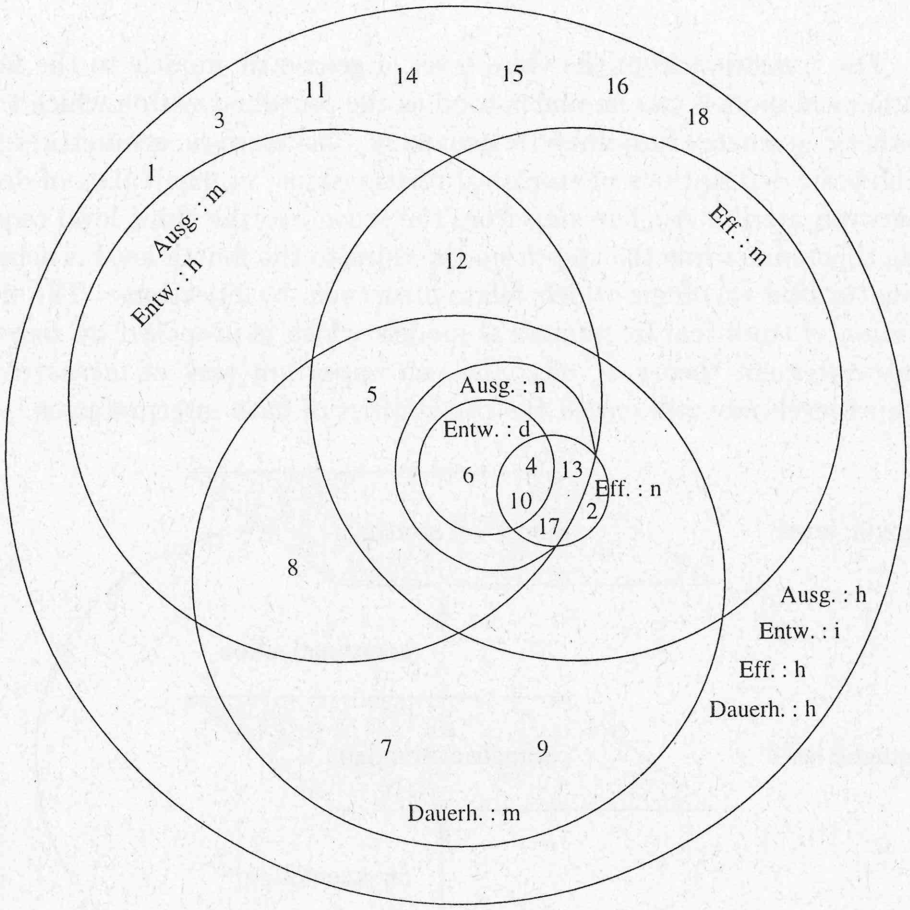
We conclude this section with a brief discussion of a data set from political sciences on international cooperations called 'regimes'. For a comparative analysis of international regimes, Kohler-Koch (1989) has established a data table part of which is shown below. In the study of international regimes the strength of regimes is an aspect of interest which is assumed to be determined by the attributes 'Ausgestaltung', 'Entwicklungsgrad', 'Effektivität', and 'Dauerhaftigkeit'. The attribute values of these attributes are ordinal in character, where the orders on the attribute values are h ('hoch') $> m$ ('mittel') $> n$ ('niedrig') and i ('implementiert') $> h$ ('handlungsanleitend') $> d$ ('deklaratorisch').

We offer two visualizations of the closure system associated to the data table concerning the strength of the 18 considered regimes. It is obvious how to read the Venn-diagram. In the line diagram the elements of $\Gamma(\bigcup_{m \in M} \mathcal{S}_m)$ are represented by little circles and the elements of a member of $\Gamma(\bigcup_{m \in M} \mathcal{S}_m)$ are all objects which can be reached from the corresponding circle by descending sequences of lines. From both diagrams one can immediately see dependencies of attributes;

for example, every regime with 'Ausgestaltungsgrad: niedrig' and 'Effektivität: niedrig' also has the attribute 'Dauerhaftigheit: niedrig'.

	Internationale Regime	Ausgestalt-ungsgrad	Entwick-lungsgrad	Effektivität	Dauer-haftigkeit
1	Cocom	h	i	h	h
2	Antarktis	h	i	n	m
3	Nonproliferation	h	i	h	h
4	Streitbeilegung	n	d	n	n
5	Menschenrechte	m	h	m	m
6	Wirt., Wiss., Technik	n	d	m	m
7	Freizügigkeit	h	i	h	m
8	Journalismus	m	h	h	m
9	KVAE	h	i	h	m
10	Nonintervention	n	d	n	n
11	Rhein	h	i	h	h
12	Nordsee	m	h	m	h
13	Mittelmeer	n	h	n	n
14	Ostsee	h	i	h	h
15	Luft	h	i	h	h
16	Ozon	h	i	h	h
17	Daten	n	h	n	n
18	Schulden	h	i	h	h

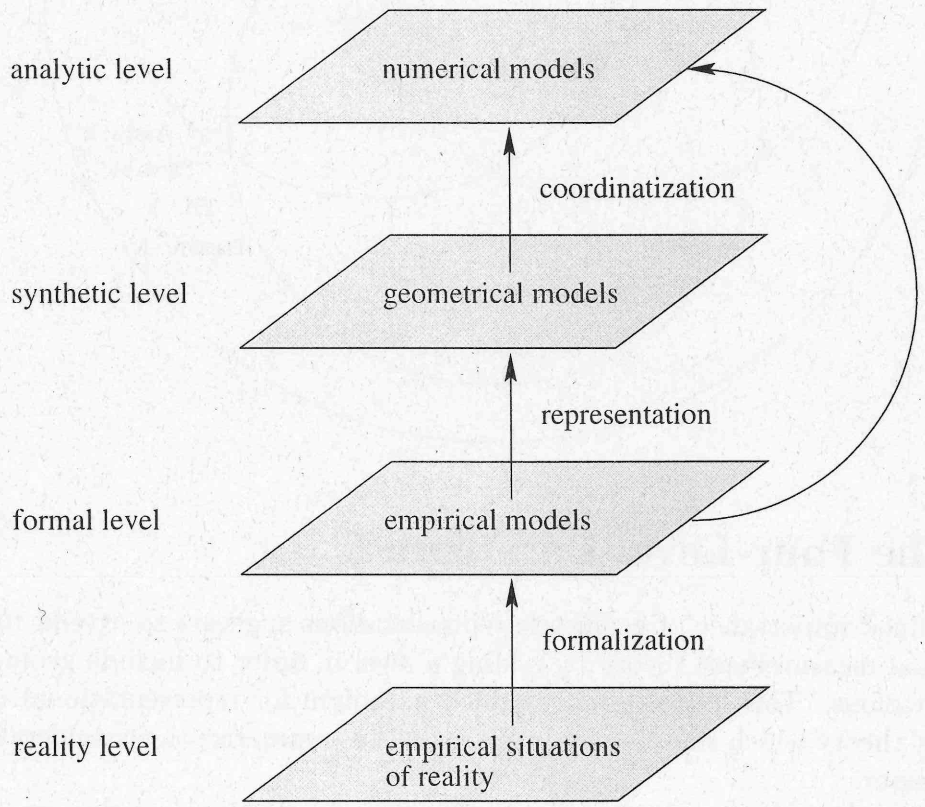




5 The Four-Level-Paradigm

The outlined importance of geometric representations suggests to extend representational measurement theory by adding a level in order to include geometric representations. This leads to an extended paradigm for representational measurement theory which shall briefly be discussed to summarize the considerations of this paper.

The idea is to distinguish four levels. The first level comprises the empirical situations of reality. In a first step of mathematization empirical situations are formalized by empirical models which are located on the second level; in particular, the mathematically viewed data sets belong to this level. The third level is the new level which emphasizes the independent role of synthetic geometric spaces as representing structures. In order to perform representations of empirical models in geometric spaces, embeddability conditions have to be found for axiomatically defined geometric structures, for instance euclidean spaces, affine spaces, geometric nets, and more general geometric spaces. A representation on the third level is often sufficient to obtain satisfying interpretations of the empirical data. The transition from the third level of geometric models to the fourth level of numerical models can be understood as the coordinatization which transforms synthetic geometry into analytic geometry. The analytic geometric spaces provide algebraic descriptions of empirical relationships, in particular, of dependencies between attributes. The step from the second to the third level requires embedding theorems, while the step from the third to the fourth level is achieved by coordinatization theorems which relate structures by bijections. The direct representation of empirical by numerical models which is proposed by representational measurement theory is, of course, an important task of measurement. But the third level may still enrich the possibilities of data interpretation.



References

[1] Ganter, B., Wille, R., Conceptual scaling. In F.S. Roberts (ed.): *Applications of combinatorics and graph theory to the biological and social sciences.* New York: Springer-Verlag, 1989, 139–167.

[2] Ganter, B., Wille, R., *Formale Begriffsanalyse: Mathematische Grundlagen.* Berlin, Heidelberg: Springer-Verlag, 1996.

[3] Jónsson, B., Lattice-theoretic approach to projective and affine geometry. In L. Henkin, P. Suppes, A. Tarski (eds.): *The axiomatic method with special reference to geometry and physics.* Amsterdam: North-Holland, 1959, 188–203.

[4] Kohler-Koch, B., Zur Empirie und Theorie internationaler Regime. In B. Kohler-Koch (ed.): *Regime in den internationalen Beziehungen.* Baden-Baden: Nomos-Verlag, 1989, 17–85.

[5] Krantz, D.H., Luce, R.D., Suppes, P., Tversky, A., *Foundations of measurement.* Vol.1, San Diego: Academic Press, 1971.

[6] Maeda, F., Lattice theoretic characterization of abstract geometries. *Journal of Science of the Hiroshima University* 1951–1952, 15, 87–96.

[7] Savage, L.J., *The foundations of statistics.* New York: Wiley, 1954.

[8] Schiffmann, H., Falkenberg, Ph., The organization of stimuli and sensory neurons. *Physiology and Behavior* 1968, 3, 197–201.

[9] Scott, D., Measurement Structures and Linear Inequalities. *Journal of Mathematical Psychology* 1964, 1, 233–247.

[10] Strahringer, S., Wille, R., Towards a structure theory for ordinal data. In M. Schrader (ed.): *Analysing and modeling data and knowledge.* Heidelberg: Springer-Verlag, 1992, 129–139.

[11] Suppes, P., Krantz, D.H., Luce, R.D., Tversky, A., *Foundations of measurement.* Vol.2, San Diego: Academic Press, 1989.

[12] Wille, R., Restructuring lattice theory: an approach based on hierarchies of concepts. In I. Rival (ed.): *Ordered sets.* Boston-Dordrecht: Reidel, 1982, 445–470.

[13] Wille, R., Wille, U., Coordinatization of ordinal structures. FB4-Preprint Nr. 1454, TH Darmstadt, 1992.

[14] Wille, R., Wille, U., Uniqueness of coordinatizations of ordinal structures. *Contribution to General Algebra 9.* Wien: Verlag Hölder-Pichler-Tempsky, 1995, 321–324.

[15] Wille, U., *Geometric representation of ordinal contexts.* Dissertation, Gießen, 1995.

[16] Wille, U., Representation of ordinal contexts by ordered n-quasigroups. *Europ. J. Combinatorics* 1996, 17, 317–333.

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